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Lower bounding techniques for frequency assignment

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Abstract

The frequency assignment problem is an NP complete problem of great importance to the radiocommunications industry. Most current solution techniques for real frequency assignment problems use heuristic algorithms to obtain suboptimal solutions in an acceptable time. By formulating the problems in terms of graph colourings, lower bounds can be obtained to assess the quality of these heuristic solutions.

Bounds based on the travelling salesman problem have proved to be successful, in some cases giving tight bounds when applied to a suitable subproblem. However, for general problems these bounds may be difficult to calculate or are far from optimal. The choice of subproblem is critical in evaluating these bounds and can also be of use in the application of the heuristic algorithms. In this paper we present a number of new and improved techniques for determining lower bounds.

1. Introduction

The practical importance of the frequency assignment problem has led many authors to propose heuristic algorithms for finding good assignments [2,3,6,12]. Lower bounds are necessary in order to assess how good these assignments are. A comprehensive account of lower bounds for the frequency assignment problem can be found in [11].

It turns out that for many practical frequency assignment problems, the Hamiltonian path (or *Travelling Salesman*) bound [11] is often tight. See [12] for optimal solutions of some well-known practical problems. However, there remain many types of practical problem for which the Hamiltonian path bound is not tight, and improved bounds are necessary. It seems that no single bound will always be tight and a variety of lower bounding techniques may be necessary to find good lower bounds for all of the

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problems that may arise in practice. In this paper we formulate a variety of techniques that can be used, and give some typical results.

Generally, lower bounds are found not from the constraint graph of the problem, but from a suitable subgraph of the constraint graph. This is often a clique or a clique with some vertices added. An alternative approach may be to use a *kernel* as defined in [11]. We modify the definition of the kernel so that it becomes unique and generalise the new definition. The subgraphs obtained may be useful for computing lower bounds, or for starting configurations for heuristic algorithms, as described in [12].

2. The frequency assignment problem

A radio communications network consists of a set of transmitters in a given region. The frequency assignment problem (FAP) is to assign each transmitter a frequency from a given set in order to satisfy some given requirement on the interference between signals. For a discussion of the formulation of the frequency assignment problem see Hale [5].

Following [11] we express the frequency assignment problem as a graph colouring problem.

Definition 1. A *constraint graph* G is a finite, simple, undirected graph in which each edge v_iv_j $(v_i, v_j \in V(G))$ has an non-negative integer label ϕ_{ij}.

Definition 2. A *frequency assignment* in a constraint graph G is a mapping $f: V(G) \to F$ (where F is a set of consecutive integers $0, \dots, K$) such that the constraints

$$|f(v_i) - f(v_j)| > \phi_{ij}$$

are satisfied for all $v_iv_j \in E(G)$. Sometimes this is referred to as a *zero-violation* assignment. If one or more of the inequalities are violated, we can refer to f as an assignment with constraint violations.

Definition 3. If K is a minimum over all zero-violation assignments then the assignment is a minimal assignment. This minimal value of K is the *minimum span* of G, denoted $\mathrm{sp}(G)$.

3. Lower bounds from the travelling salesman problem

Hamiltonian paths were first used in [10] to derive bounds for the frequency assignment problem. To calculate the bounds we first construct from G a weighted complete graph G' on the vertices of G. The weight c_{ij} of each edge v_iv_j of G' is given by

$c_{ij} = 0$ if v_iv_j is not an edge of G,
$c_{ij} = \phi_{ij} + 1$ if edge v_iv_j has label ϕ_{ij} in G $(\phi_{ij} = 0, 1, \dots)$.

Let $H(G')$ be the weight of a minimum cost Hamiltonian path in G' and $S(G')$ denote the weight of a minimum cost spanning tree of G'. If the Hamiltonian path is $\{v_{i_1},\dots,v_{i_n}\}$ then we can get a frequency assignment (possibly with some constraint violations between non-consecutive vertices on the path) by setting

$$\begin{aligned} f(v_{i_1}) &= 0, \\ f(v_{i_j}) &= f(v_{i_{j-1}}) + c_{i_{j-1}i_j} \quad \text{for } j=2,\dots,n. \end{aligned}$$

Proposition 1 (Smith and Hurley [11]). *If G is a constraint graph then*

$$\mathrm{sp}(G) \geqslant H(G') \geqslant S(G').$$

Often the bounds given by Proposition 1 are weak when applied to the whole constraint graph. Typically, G' has many edges with weight 0, and in this case the bound is often close to 0. However, applying Proposition 1 to some subgraph of G' also gives a valid lower bound as the minimal span of a subgraph cannot be greater than that of the whole graph. In particular, a better bound can usually be obtained by applying Proposition 1 to some clique of G.

Definition 4. A *level-p clique* of G is a complete subgraph in which every edge has label at least p, which is not contained in any larger such complete subgraph.

Proposition 2 (Smith and Hurley [11]). *If C_p is a level-p clique of a constraint graph G then*

$$\mathrm{sp}(G) \geqslant H(C_p') \geqslant S(C_p') \geqslant (p+1)(|V(C_p)|-1).$$

In practice, applying Proposition 1 to a subgraph of the constraint graph consisting of a clique together with some further vertices can also give good results. As described in [12], the additional vertices can be chosen by assigning the current subgraph and attempting to extend the assignment to the whole constraint graph. Vertices which are hard to assign without using further frequencies are added to the subgraph, and the bounds are recalculated for the new subproblem. This process can be repeated until either an optimal assignment is found or the lower bounds for the subgraphs begin to decrease.

In general, exact methods for calculating $H(G')$ (which is equivalent to solving an open symmetric travelling salesman problem) are impractical. A branch-and-bound algorithm of Volgenant and Jonker [14] has been used successfully for some problems but the available software [13] is limited to a maximum of 250 vertices and is not guaranteed to find a solution for problems below this size. For many frequency assignment problems this is too restrictive and so lower bounds on $H(G')$ must be used. These can be obtained by formulating the travelling salesman problem as an integer program. Let the graph G_0' be formed from G' by the addition of a vertex v_0 joined by an edge of weight 0 to each vertex of G'. Then $H(G')$ is equal to the solution of the following integer program for the closed symmetric travelling salesman problem (TSP):

$$\text{Minimize} \quad \sum_{v_iv_j \in E(G_0')} c_{ij}x_{ij} \tag{1}$$

$$\text{subject to} \quad \sum_{j:v_iv_j \in E(G_0')} x_{ij} = 2, \quad v_i \in V(G_0'), \tag{2}$$

$$\sum_{v_i \in S, v_j \in V(G_0')\setminus S} x_{ij} \geqslant 2, \quad S \subset V(G_0'), \tag{3}$$

$$x_{ij} \in \{0,1\}, \quad v_iv_j \in E(G_0'). \tag{4}$$

The following two methods can be used to give lower bounds for $H(G')$:

- Relaxing integrality constraint (4) to

  $$0 \leqslant x_{ij} \leqslant 1, \quad v_iv_j \in E(G_0') \tag{5}$$

  gives the linear (LP) relaxation of the integer program, which has been shown for random graphs [7] to be on average within 1% of the integer solution.
- Removing constraints (3) (the so-called subtour elimination constraints) gives an integer program for a minimum cost perfect two matching in G_0'. We denote this integer program by (2Mat) and we denote the value of a solution to the program by $M_2(G_0')$. There exists an algorithm for finding such a two matching which is guaranteed to terminate in $O(|V(G)|^2\,|E(G)|)$ [8]. A lower bound for $M_2(G_0')$ can be obtained by replacing integrality constraint (4) by constraint (5).

Computational experience for constraint graphs derived from realistic frequency assignment problems suggests that subtour elimination constraints have little effect on the tightness of the bound.

4. Adding frequency assignment constraints

In general, bounds obtained from a TSP solution are not tight as such a solution ignores constraints between non-consecutive vertices in the Hamiltonian path. By adding further constraints to the integer program of the TSP it is possible to ensure that the frequency separation between non-consecutive vertices on the path is sufficient. This integer program cannot be solved in reasonable time. However, by considering a subset of the new constraints and relaxing the integrality constraint, solutions can be obtained in an acceptable time which are better than the TSP solutions.

In order to add extra frequency assignment constraints we associate a non-negative integer variable e_{ij} with each edge v_iv_j of the constraint graph G. We want to choose the e_{ij} so that when we construct an assignment from a Hamiltonian path $\{v_{i_1},\dots,v_{i_n}\}$ by setting

$$f(v_{i_1}) = 0,$$
$$f(v_{i_j}) = f(v_{i_{j-1}}) + c_{i_{j-1}i_j} + e_{i_{j-1}i_j} \quad \text{for } j = 2,\dots,n,$$

the assignment will have no constraint violations. Then constraints between consecutive vertices on the Hamiltonian path no longer have to be met exactly, allowing constraints between non-consecutive vertices to be satisfied. In this case the value e_{ij} is referred to as the *excess* on the edge $v_i v_j$.

To formulate the frequency assignment constraints we make the following definitions. If P is a path $v_{i_1}, v_{i_2}, \ldots, v_{i_k}$ with edge set $E(P)$, then let $X_P = x_{i_1 i_2} + \cdots + x_{i_{k-1} i_k}$ and $E_P = e_{i_1 i_2} + \cdots + e_{i_{k-1} i_k}$. Define the *deficit* of P as

$$d(P) = c_{i_1 i_k} - (c_{i_1 i_2} + \cdots + c_{i_{k-1} i_k}).$$

Let $\mathscr{P}(G')$ be the set of paths P of G' with $d(P) > 0$. Then if $P \in \mathscr{P}(G')$ we require that

$$X_P - (|E(P)| - 1) \leqslant \frac{E_P}{d(P)}.$$

If $X_P \leqslant (|E(P)| - 1)$ then E_P is unconstrained. If $X_P = |E(P)|$ (that is all edges of P are included in the Hamiltonian path) then the total excess on P must be at least as large as the deficit of P to ensure that the constraint between the end vertices of P is satisfied. This gives the following integer programming formulation of the FAP:

$$\text{Minimize} \quad \sum_{v_i v_j \in E(G'_0)} c_{ij} x_{ij} + \sum_{v_i v_j \in E(G')} e_{ij} \tag{6}$$

$$\text{subject to} \quad \sum_{j : v_i v_j \in E(G'_0)} x_{ij} = 2, \quad v_i \in V(G'_0), \tag{7}$$

$$\sum_{v_i \in S,\, v_j \in V(G'_0) \setminus S} x_{ij} \geqslant 2, \quad S \subset V(G'_0), \tag{8}$$

$$d(P) X_P - d(P)(|E(P)| - 1) - E_P \leqslant 0, \quad P \in \mathscr{P}(G'), \tag{9}$$

$$x_{ij} \in \{0, 1\}, \quad v_i v_j \in E(G'_0), \tag{10}$$

$$e_{ij} \in \{0, 1, \ldots, c_{\max}\}, \quad v_i v_j \in E(G'), \tag{11}$$

where $c_{\max} = \max_{i,j} c_{ij}$.

This integer program can be used to derive lower bounds for the FAP as follows:

4.1. Linear programming relaxations

By replacing integrality constraints (10) and (11) by

$$0 \leqslant x_{ij} \leqslant 1, \quad v_i v_j \in E(G'_0), \tag{12}$$

$$0 \leqslant e_{ij} \leqslant c_{\max}, \quad v_i v_j \in E(G'), \tag{13}$$

we obtain a linear programming relaxation of the FAP. By also omitting constraint (8) we obtain a linear programming relaxation of 2Mat with extra frequency assignment constraints (2Mat+FAP). The number of constraints (9) can be impractical for many choices of subgraph. This can be remedied by only considering paths of length two. In this case it is generally acceptable to identify all possible violating paths by a complete search of the subgraph. For many choices of subgraph this relaxation is strong. For example, if G has a maximum edge-label of 4 (i.e. $\max c_{ij}=5$) and C is a level-1 clique, then $\mathscr{P}(C')$ can only contain paths of length two. Clearly, if additional vertices are added to the clique, the effect of the restriction may become significant.

4.2. Lagrangean relaxation

Using Lagrangean relaxation [1,4] we can relax constraint (9) into the objective function to give the cost function

$$L' = \sum_{v_iv_j\in E(G_0')} c_{ij}x_{ij} + \sum_{v_iv_j\in E(G')} e_{ij} + \sum_{P_k\in\mathscr{P}(G')} \mu_k[d(P_k)X_{P_k} - d(P_k)(|E(P_k)|-1) - E_{P_k}].$$

This gives the following Lagrangean lower bound program:

$$\begin{aligned} &\text{Minimize} && L' \\ &\text{subject to} && x_{ij}\text{'s form a Hamiltonian path,} \\ &&& x_{ij}\in\{0,1\}, \quad v_iv_j\in E(G_0'), \\ &&& e_{ij}\in\{0,1,\ldots,c_{\max}\}, \quad v_iv_j\in E(G'). \end{aligned}$$

For any choice of multipliers $\mu_1,\ldots,\mu_{|\mathscr{P}(G')|}\geqslant 0$, the solution to this program provides a lower bound for the original problem.

If $\mu_1,\ldots,\mu_k$ are all the multipliers corresponding to constraints containing e_{ij}, then the term containing e_{ij} in the cost function is

$$e_{ij} - (\mu_1+\cdots+\mu_k)e_{ij}.$$

Thus provided we choose multipliers such that $(\mu_1+\cdots+\mu_k)\leqslant 1$, we may assume $e_{ij}=0$. If this property is satisfied for all e_{ij} then the minimization problem for any such set of multipliers is equivalent to solving the TSP in G' with modified edge weights. That is, the program

$$\begin{aligned} &\text{minimize} && \sum_{v_iv_j\in E(G_0')} c_{ij}x_{ij} + \sum_{P_k\in\mathscr{P}(G')} \mu_k[d(P_k)X_{P_k} - d(P_k)(|E(P_k)|-1)] \\ &\text{subject to} && x_{ij}\text{'s form a Hamiltonian path,} \\ &&& x_{ij}\in\{0,1\}, \quad v_iv_j\in E(G_0'). \end{aligned}$$

Thus, we would ideally like to solve the Lagrangean dual program:

$$\max_{\mu_i}\left\{\begin{array}{ll}\text{Minimize} & \sum\limits_{v_iv_j\in E(G_0')} c_{ij}x_{ij} \\ & +\sum\limits_{P_k\in\mathscr{P}(G')} \mu_k[d(P_k)X_{P_k}-d(P_k)(|E(P_k)|-1)] \\ \text{subject to} & x_{ij}\text{'s form a Hamiltonian path,} \\ & x_{ij}\in\{0,1\},\quad v_iv_j\in E(G_0')\end{array}\right\}$$

to obtain the best lower bound possible given our assumption on μ_i.

5. Other lower bounding methods

5.1. Odd cycle bound

The following elementary result, which is stated without proof, has been found to give a better bound than methods described previously for certain types of radio links frequency assignment problem with large edge weights:

Proposition 3. *Let G be a constraint graph and G' be the modified constraint graph derived from G. If $\mathscr{C}$ is a cycle of odd length contained in G' in which each edge has a non-zero constraint then*

$$\mathrm{sp}(G)\geqslant\max(c_M,c_1+c_2)$$

where c_M is the maximum constraint on $\mathscr{C}$ and c_1,c_2 are the two smallest constraints on $\mathscr{C}$.

5.2. Branch and bound

Given an constraint graph G, the set of paths $\mathscr{P}(G')$ can be used as the basis for a branch-and-bound method to derive lower bounds or optimal assignments for the frequency assignment problem. If $P\in\mathscr{P}(G')$, $(d(P)>0)$ then in any zero-violation assignment of G at least one edge v_iv_j of P must satisfy

$$|f(v_i)-f(v_j)|-c_{ij}>0.$$

Therefore, if G_{ij} is the graph formed from G by
- if $v_iv_j\notin E(G)$ adding an edge v_iv_j with $\phi_{ij}=0$, or
- if $v_iv_j\in E(G)$ replacing ϕ_{ij} by $\phi_{ij}+1$,

then

$$\mathrm{sp}(G)\geqslant\min_{v_iv_j\in E(P)}\mathrm{sp}(G_{ij}).$$

A branch-and-bound method based on this result may require the computation of a very large tree to give an exact solution. However, lower bounds can still be obtained by exploring part of the tree. Some results are presented in Section 6.

6. Results

The following example demonstrates the use of these lower bounding techniques on a small frequency assignment problem typical of a military radio links problem. This problem consists of only 38 transmitters, so the value of sp(G) can be found using an exact algorithm. This allows the lower bounds to be compared with the exact value of sp(G). The constraint graph has a level-0 clique of 12 vertices.

Bounding method applied to level-0 clique	Bound
$S(G')$	15
$H(G')$	20
2Mat LP relaxation	20
Lagrangean relaxation	21
2Mat+FAP LP	22
Branch and bound with a TSP bound at each node	22
Branch and Bound with 2Mat+FAP LP bound at each node	23
Exact solution	24

The bounding techniques were implemented as follows:

$S(G')$: Calculated using Prims algorithm [9].

$H(G')$: Calculated using software for the travelling salesman problem [13].

2Mat LP *and* **2Mat+FAP LP:** Calculated using a commercial linear programming package. Frequency assignment constraints were only added for paths $P \in \mathscr{P}$ with $|E(P)| = 2$ for speed and memory reasons, which may reduce the lower bound.

Lagrangean relaxation: Uses the travelling salesman problem software [13] to calculate the solution for each choice of multipliers. The best-solution found has nine non-zero multipliers out of a total of 65 paths of two edges.

Branch and bound: A total of 27 nodes were explored to achieve the bound.

Exact solution: Calculated using a back-tracking algorithm [6].

It may be possible to improve the results of branch and bound and Lagrangean relaxation further by exploring more of the tree or by a better choice of multipliers, respectively. Lagrangean relaxation was further tested on several other problems. In each case it was outperformed by the linear program 2Mat+FAP LP, yet took considerably more time.

In Tables 1–3 we present more results comparing the results from 2Mat LP and 2Mat+FAP LP with TSP solutions obtained using the travelling salesman problem software. The TSP software is not guaranteed to find a solution for all problems; if no solution is found upper and lower bounds from the software are given. Table 1 gives bounds for cliques from cellular frequency assignment problems and Table 2 shows

Table 1
Bounds evaluated for cliques

Example	Number of transmitters	$S(G')$	$H(G')$			2Mat LP
			Exact [13]	Lower bound	Upper bound	
Cell1	140	159	—	159	177	**177**
Cell2	180	179	—	179	218	**218**
Cell3	275	351	**426** [a]	—	—	**426**
Cell4	360	359	— [b]	— [b]	— [b]	**370**

[a] Could not be evaluated because of software restrictions. These values were obtained by inspection, see [12].
[b] Could not be evaluated because of software restrictions.

Table 2
Bounds evaluated for cliques

Example	Number of transmitters	$S(G')$	$H(G')$			2Mat LP	2Mat+FAP LP
			Exact [13]	Lower bound	Upper bound		
Links1	26	54	**67**	—	—	63	64
Links2	34	69	75	—	—	75	**76**
Links3	45	62	75	—	—	75	**77**
Links4	59	122	—	125	128	126	**130**
Links5	79	103	—	126	128	125	**129**
Links6	99	126	—	153	157	153	**156**

Table 3
Bounds evaluated for a clique with additional vertices

Example	Number of transmitters	$S(G')$	$H(G')$			2Mat LP	2Mat+FAP LP
			Exact [13]	Lower bound	Upper bound		
Links7	70	149	171	—	—	171	**172**
Links7+5	75	144	—	174	177	174	**178**
Links7+10	80	138	—	177	178	176	**181**
Links7+15	85	133	—	175	176	175	**179**
Links7+25	95	108	—	170	171	169	**175**
Links7+35	105	87	—	161	165	160	**166**

bounds evaluated on cliques from military radio links problems. Table 3 shows how the bound for a clique in a radio links problem varies as additional vertices are added to the clique. In each of Tables 1–3 the best bound is shown in bold face.

Heuristic algorithms can be used to compute assignments of minimal or near minimal span [6]. Generally, bounds computed by the techniques we have described either equal the minimum span [12] or are close to the minimum span.

Note that the upper and lower bounds returned by the TSP software may or may not be equal. However, by comparing 2Mat LP and 2Mat+FAP LP with the computed upper bound on $H(G')$ provided by the TSP software, a worst-case comparison can be made. For both links and cellular problems, 50 subgraphs were selected. The subgraphs were cliques with a small number of randomly chosen additional vertices. On average, 2Mat LP was at most 0.9% and 0.4% less than the upper bound for $H(G')$, respectively. 2Mat+FAP LP showed an average improvement of at least 3.1% for links problems and 0.2% for cellular problems. However, for other cellular problems for which the TSP bound could be exactly calculated, 2Mat+FAP LP showed an improvement of up to 15%.

7. Kernels

The following definitions are taken from [11].

Definition 5. Suppose that for a given vertex $v_i \in V(G)$ there exists a non-adjacent vertex $v_j \in V(G)$ such that for each edge v_iv_k incident with v_i, v_jv_k is an edge of G with $\phi_{jk} \geqslant \phi_{ik}$. Vertex v_j is said to *cover* v_i, and v_i is called a *covered* vertex.

Definition 6. Suppose that J is an estimate of the span of G. Let

$$P(v_i) = \sum_{v_k \in \Gamma(v_i)} (2\phi_{ik} + 1).$$

If

$$P(v_i) < J + 1,$$

then v_i is called a *deficient* vertex.

Deficient vertices are vertices v_i for which a frequency must be available if $G - v_i$ is assigned using a set $\{0, 1, 2, \ldots, J\}$ of frequencies. In [11] a subgraph of G called a kernel was defined in terms of deficient and covered vertices. However, such a kernel is not unique as arbitrary choices of covered vertices can sometimes be made. Covered vertices rarely occur in frequency assignment problems. Thus, we will define the kernel in terms of deficient vertices alone and show that with the new definition the kernel is unique.

Suppose that we are attempting to assign G with a set $\{0, 1, 2, \ldots, J\}$ of frequencies. A sequence

$$G = G_0, v_0, G_1, v_1, \ldots, v_{i-2}, G_{i-1}, v_{i-1}, G_i, v_i, G_{i+1}, \ldots, G_m$$

can be constructed as follows:

Given a graph G_i in the sequence, choose a vertex v_i which is deficient in G_i. Then $G_i - v_i = G_{i+1}$. Continue this until no further deficient vertices can be chosen. Find (if possible) an assignment of G_m which uses frequencies from the set $\{0, 1, 2, \ldots, J\}$.

Now work backwards through the sequence adding in the vertices in reverse order and assigning them until an assignment of G using frequencies from the set $\{0,1,2,\dots,J\}$ is obtained. If $J \leqslant \mathrm{sp}(G)$, a frequency is always available for a deficient vertex from $\{0,1,2,\dots,J\}$. G_m is referred to as the *kernel* and the problem of assigning G_m is the *kernel problem*.

If J is chosen to be an overestimate of the span of G and G_m can be assigned using frequencies from the set $\{0,1,2,\dots,K\}$ with $(K<J)$ then it is not always possible to assign G_{i-1} directly using the same span as G_i. However, if $J-K$ is small, an assignment of G_m may often be extended heuristically to an assignment of $G_0=G$. In this case it is convenient to refer to the kernel as the *J-kernel*.

Proposition 4. *For a given constraint graph G and choice of J, the J-kernel is unique.*

Proof. Assume that the J-kernel is not unique and, without loss of generality, assume that K_1 and K_2 are J-kernels with $K_1\backslash K_2$ non-empty. Let $\sigma_1=\{v_{i_1},v_{i_2},\dots,v_{i_p}\}$ be the sequence of vertices deleted to obtain K_1 from G and $\sigma_2=\{v_{j_1},v_{j_2},\dots,v_{j_q}\}$ be the sequence of vertices deleted to obtain K_2 from G. Let σ_3 be the subsequence of σ_2 consisting of vertices of σ_2 not in σ_1. Consider the sequence $\sigma_1\sigma_3$. Let $w\in\sigma_3$ and suppose that all previous vertices in $\sigma_1\sigma_3$ have been deleted. Then $P(w)$ with respect to the sequence $\sigma_1\sigma_3$ is no greater than $P(w)$ with respect to the sequence σ_2. Thus w can be deleted. Repeating this for all vertices in σ_3 contradicts the assumption that K_1 and K_2 are J-kernels with $K_1\backslash K_2$ non-empty. $\square$

The J-kernel can be used both for finding assignments using the method of [12], and for deriving lower bounds. When finding assignments, if J has been chosen equal to the span of G, an assignment of the J-kernel of span J clearly has an extension to an assignment of G of span J, and this assignment is normally easy to find heuristically. In fact, when J is significantly larger than the span of G, a good assignment of the J-kernel may still have an extension to an assignment of G of the same span J, and this assignment can be found heuristically in many examples [12]. The J-kernel, with a suitable choice of J, may also be a useful subgraph to use for finding lower bounds. It has the advantage for larger problems that it is easier to compute than a clique or a clique with vertices added. However, as it may not be complete, it is more often useful when the bound makes use of added frequency assignment constraints (2Mat+FAP).

8. Conclusion

The Hamiltonian path bound has previously been used to find the minimum span for practical frequency assignment problems exactly. In other cases the Hamiltonian path bound is either difficult to compute or falls some way short of the best lower bound achievable. The bounds we have presented address both of these issues. However, no one technique is of universal applicability.

For the most difficult problems, finding a tight lower bound requires a subgraph substantially larger than the cliques and cliques with a small number of vertices added that we have used. For such subgraphs, the goal is to find a bound that is computable in reasonable time and does not tend to zero as the size of the subgraph used increases. The 2Mat+FAP LP technique we have presented is a significant step in this direction, but the development of this technique to handle larger and more difficult problems remains a substantial challenge.

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